Consistent lattice Boltzmann schemes for the Brinkman model of porous flow and infinite Chapman-Enskog expansion

Irina Ginzburg*

Antony Regional Centre, Centre de Recherche pour l'Ingénierie de l'Agriculture et de l'Environment (Cemagref), HBAN, Parc de Tourvoie—Boîte Postale 44, 92163 Antony cedex, France

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We show that a consistent modeling of porous flows needs at least one free collision relaxation rate to avoid a nonlinear dependency of the numerical errors on the viscosity. This condition is necessary to get the viscosity-independent permeability from the Stokes flow and to parametrize properly (with nondimensional physical numbers) the lattice Boltzmann Brinkman schemes. The two-relaxation-time (TRT) collision operator controls all coefficients of the higher-order corrections in steady solutions with a specific combination of its two collision rates, a possibility lacking for the Bhatnagar-Gross-Krook (BGK)-based single-relaxation-time schemes. The analysis is based on exact recurrence equations of the evolution equation and illustrated for the exact solutions of the Brinkman scheme in simply oriented parallel and diagonal channels. The apparent viscosity coefficient of the TRT Stokes-Brinkman scheme in arbitrary flow is only approximated. The compatibility of one-dimensional arbitrarily rotated flows with the nonlinear (Navier-Stokes) equilibrium is examined. An explicit dependency for all coefficients on the relaxation rates is presented for the infinite steady state Chapman-Enskog expansion.

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I. INTRODUCTION

The description of viscous flow inside a mixed fluid-porous system is needed for many technological problems, such as, for example, permeability studies for materials with distinct pore size length scales or fractures [1–3], liquid composite molding [4], or flows through filters [5]. The Brinkman equation [6], a semiempirical viscous modification of Darcy's law for the volume-averaged pressure $\bar{P}$ and velocity $\vec{u}$,

$$\vec{\nabla}\cdot\vec{u}=0,$$

$$\nu\mathbf{K}^{-1}\vec{u}+\frac{1}{\rho_0}\vec{\nabla}\bar{P}-\vec{g}=\frac{\nu_e}{\phi}\Delta\vec{u}, \tag{1}$$

aims to account for the presence of the solid boundaries in Darcy (small-pore) flows, combined with a Stokes description for free (large-pore) fluid. Darcy's law is recovered when the velocity derivatives are sufficiently small, i.e., the scale of velocity variation is much larger than $O(\sqrt{\|K\|})$. The Brinkman equation, theoretically justified in [9] via the volume averaging of the Stokes equation, is also used as a transmission condition (e.g., [10–13]) between a porous flow and a free fluid. The effective value of the Brinkman viscosity coefficient, hereafter $\frac{\nu_e}{\phi}$ with ϕ as the porosity of the medium, is still a subject of theoretical and numerical investigations (e.g., [1,14–16]).

A uniform numerical scheme can be designed for solving the microscopic and macroscopic porous flow in the frame of the lattice Boltzmann equation (LBE) method [17,18]. The Bhatnagar-Gross-Krook (BGK)-Brinkman models [3,4,7,19–21] incorporate the Darcy or Forchheimer resistance force into the single-relaxation-time (BGK) hydrodynamic model [8]. This paper is inspired by the recent work by Nie and Martys [7], where a discrepancy between the apparent viscosity coefficient and the predictions of the second-order Chapman-Enskog analysis, traditionally used to derive the macroscopic equations following the pioneering work by Frish *et al.* [22], is demonstrated for parallel and diagonal Brinkman channel flow.

The present work shows that this discrepancy is caused by a coupling of the second- and higher-order coefficients related to the spatial variation of the resistance forcing, proportional to the local velocity. We show that all higher-order coefficients are set by a specific (called "magic") collision combination of two relaxation parameters of the two-relaxation-time (TRT) operator [23–26]. It becomes possible to arrange them as an apparent correction of the viscosity coefficient, but only for flow parallel to one of the lattice axes. The specific values of the magic combination allow then to fit the channel velocity either to the exact (exponential) solution of the Brinkman equation or to the exact solution of the finite-difference discretization scheme.

This work extends the analysis [26,27] of the parametrization properties of the steady numerical solutions for the hydrodynamic and advection-diffusion equations to the TRT Stokes-Brinkman equation. Based on recurrence equations [27] derived from the evolution operator and without the help of the Chapman-Enskog expansion, this analysis shows that the steady Stokes flow is linear with respect to the ratio of the applied forcing to the applied kinematic viscosity, and that the Navier-Stokes-type equations are controlled, *for all orders*, by nondimensional physical numbers (such as the Reynolds and Froude numbers, in the incompressible regime, plus the Mach number, in the compressible regime) provided that the specific combinations of relaxation parameters associated with the symmetric and antisymmetric collision modes are set. We mean here that only when the fixed values are

*irina.ginzburg@cemagref.fr

prescribed for them, then the nondimensional solutions of the LB schemes *on the same grid* are identical for any variation of the physical parameters governed by the nondimensional physical numbers.

These findings provide the principal elements to explain why a prescribed value for the magic parameters fixes a viscosity-independent permeability for different types of porous structures (e.g., [27–29]). The present work extends this property for the TRT Stokes-Brinkman schemes and gives the necessary conditions to avoid a nonlinear dependency of their apparent transport coefficients on the viscosity. The magic parameter is proportional to the square of the viscosity coefficient for the single-relaxation-time model. That is why the BGK-based schemes cannot avoid a nonlinear dependency of their higher-order corrections on the viscosity, demonstrated by the exact solution [7], whatever boundary schemes are applied.

The Chapman-Enskog approach reduces to a specially arranged summation of the truncated Taylor series for steady solutions. A leading-order discrepancy in the apparent viscosity coefficient is revealed already by the second-order Chapman-Enskog analysis when the nonlinear spatial variation of the forcing is taken into account. *Infinite* expansion satisfies the recurrence equations for an arbitrary (sufficiently smooth) equilibrium function and the most general linkwise collision operator; the stationary macroscopic bulk equations then coincide in both approaches. The TRT operator presents a common subclass of the link (L) model [23,25] and multiple-relaxation-times (MRT) models [17,18,30–34]. All solutions derived for the TRT operator are valid for the MRT models, taking one relaxation value for all their symmetric (and one for all antisymmetric) collision modes.

The paper is organized as follows. The L and TRT models and their recurrence and exact steady equations are presented in Sec. II. The microscopic and macroscopic solutions of the TRT scheme for the Stokes-Brinkman channel flow are constructed in Sec. III and Appendix A. The possibilities of reaching the one-dimensional velocity and constant pressure distributions in arbitrarily rotated channels based on the Navier-Stokes-type equilibrium function are examined in Appendix B and summarized in Sec. III D. The steady solution for all coefficients of the infinite Chapman-Enskog expansion is derived in Sec. IV and Appendix C. Section V concludes the paper.

II. EVOLUTION, RECURRENCE, AND MACROSCOPIC EQUATIONS

A. The L model

We assume an equidistant d-dimensional computational mesh $\{\vec{r}\}$ where the velocity vectors $\{\vec{c}_q\}$ interconnect the grid nodes. The velocity set contains Q vectors: one zero, $\vec{c}_0=\vec{0}$, for the immobile population, and $Q-1$ nonzero ones, $\vec{c}_q=\{c_{q\alpha},\alpha=1,\ldots,d\}$, for the moving populations. Cubic velocity sets [8] with two moving classes are assumed, e.g., two-dimensional models with nine velocities ($d2Q9$), and three-dimensional models with 15 and 19 velocities, $d3Q15$ and $d3Q19$, respectively. Each nonzero velocity vector has a diametrically opposite one. Below we refer to a pair of antiparallel velocities $(\vec{c}_q,\vec{c}_{\bar{q}})$ as a *link*. The unknown variables of the scheme at the node $\vec{r}$ and time t are the components of the Q-dimensional population vector $\boldsymbol{f}(\vec{r},t)$, $\boldsymbol{f}=\{f_0,(f_q,f_{\bar{q}}),q=1,\ldots,\frac{Q-1}{2}\}$. The combinations $\psi_q^+=\psi_{\bar{q}}^+=\frac{1}{2}(\psi_q+\psi_{\bar{q}})$ and $\psi_q^-=-\psi_{\bar{q}}^-=\frac{1}{2}(\psi_q-\psi_{\bar{q}})$ are referred to as the symmetric and antisymmetric components, respectively, for any link pair $(\psi_q,\psi_{\bar{q}})$: $\psi_q=\psi_q^++\psi_q^-$, $\forall\ q$. We set $\psi_0^+=\psi_0^+=\psi_0$ and $\psi_0^-=\psi_0^-=0$ for the immobile population. A pair of collision eigenvalues $\{\lambda_q^+,\lambda_q^-\}$ governs the relaxation of the symmetric and antisymmetric nonequilibrium components, respectively. It is noted that the eigenvalues are equal for two antiparallel velocities $\lambda_q^+=\lambda_{\bar{q}}^+$ and $\lambda_q^-=\lambda_{\bar{q}}^-$. Prescribing for each link the equilibrium and source components $e_q^\pm$ and $\{S_q^\pm\}$, respectively, the update rule of the L model reads

$$f_q(\vec{r}+\vec{c}_q,t+1)=\tilde{f}_q(\vec{r},t),\quad q=0,1,\ldots,Q-1,$$

$$\tilde{f}_0(\vec{r},t)=f_0(\vec{r},t)+g_0,$$

$$\tilde{f}_q(\vec{r},t)=f_q(\vec{r},t)+g_q^++g_q^-,\quad q=1,\ldots,\frac{Q-1}{2},$$

$$\tilde{f}_{\bar{q}}(\vec{r},t)=f_{\bar{q}}(\vec{r},t)+g_q^+-g_q^-,\quad \vec{c}_{\bar{q}}=-\vec{c}_q,\quad \bar{q}=1,\ldots,\frac{Q-1}{2},$$

$$g_q^\pm=G_q^\pm+S_q^\pm,\quad G_q^\pm=\lambda_q^\pm n_q^\pm,\quad n_q^\pm=(f_q^\pm-e_q^\pm).\qquad(2)$$

The eigenvalues are taken inside the interval $]-2,0[$ of linear stability. The so-called magic combinations $\{\Lambda_q^{eo}\}$ are the products of the positive eigenvalue functions Λ_q^+ and Λ_q^-,

$$\Lambda_q^{eo}=\Lambda_q^+\Lambda_q^-,\quad \Lambda^\pm=-\left(\frac{1}{2}+\frac{1}{\lambda_q^\pm}\right),\quad \forall\ q=1,\ldots,Q-1.\qquad(3)$$

For the sake of simplicity of the algebraic expressions, the source quantities $\frac{-S_q^\pm}{\lambda^\pm}$ are put hereafter into the equilibrium. The recurrence equations [26,27] represent the linear combinations of four evolution equations (2) along one link: from bulk grid node $\vec{r}$ to grid nodes $\vec{r}\pm\vec{c}_q$ and back. At steady state, $g_0^+=0$ and $\{g_q^\pm(\vec{r}),\ q=1,\ldots,Q-1\}$ satisfies four recurrence equations:

$$g_q^\pm(\vec{r})=\left[\bar{\Delta}_q e_q^\mp-\Lambda_q^\mp\Delta_q^2 e_q^\pm+\left(\Lambda_q^{eo}-\frac{1}{4}\right)\Delta_q^2 g_q^\pm\right](\vec{r}),\qquad(4)$$

$$(\Delta_q^2 e_q^\pm-\Lambda_q^\pm\Delta_q^2 g_q^\pm-\bar{\Delta}_q g_q^\mp)(\vec{r})=0,\qquad(5)$$

where, $\forall\ \psi$,

$$\bar{\Delta}_q\psi(\vec{r})=\frac{1}{2}[\psi(\vec{r}+\vec{c}_q)-\psi(\vec{r}-\vec{c}_q)],$$

$$\Delta_q^2\psi(\vec{r})=\psi(\vec{r}+\vec{c}_q)-2\psi(\vec{r})+\psi(\vec{r}-\vec{c}_q).\qquad(6)$$

It is shown in [27] that the solution to the two first equations (4) and one of the two equations (5) satisfies the remaining equation.

B. The TRT model

The TRT model sets all λ_q^+ equal to λ^+ and all λ_q^- equal to λ^-. This model has only one free magic parameter, called Λ_{eo} hereafter:

$$\Lambda_q^{eo} = \Lambda_{eo} = \Lambda_o\Lambda_e, \quad \Lambda_q^- = \Lambda_o = -\left(\frac{1}{2} + \frac{1}{\lambda^-}\right),$$
$$q = 1, \ldots, Q-1,$$
$$\Lambda_q^+ = \Lambda_e = -\left(\frac{1}{2} + \frac{1}{\lambda^+}\right), \quad q = 0, \ldots, Q-1. \tag{7}$$

When λ^+ is fixed via the kinematic viscosity, which is proportional to Λ_e, or when λ^- is fixed via the diffusion coefficient, proportional to Λ_o, the TRT operator can maintain Λ_{eo} at any positive value with the help of the second (free) eigenvalue. The BGK operator represents a subclass of the TRT operators, with $\lambda^- = \lambda^+ = -\frac{1}{\tau}$, $\Lambda_{eo} = (\tau - \frac{1}{2})^2$, and no free relaxation parameter. Using one common eigenvalue λ^+ for all the symmetric components enables the TRT operator to obey the mass conservation law based on the mass-conserving equilibrium functions $\sum_{q=0}^{Q-1} e_q^+ = \sum_{q=0}^{Q-1} f_q = \rho$. One common eigenvalue λ^- for all the antisymmetric components enables the TRT model to match the hydrodynamic equations with the momentum-conserving equilibrium functions $\sum_{q=1}^{Q-1} e_q^- \vec{c}_q = \sum_{q=1}^{Q-1} f_q \vec{c}_q = \vec{J}$.

For Stokes and Stokes-Brinkman flows driven by a pressure drop and/or a forcing, the momentum-conserving equilibrium function plus the force quantity $-\frac{F_q^\star}{\lambda^-}$ take the form

$$e_q^+(\vec{r},t) = P_q^\star, \quad P_q^\star = t_q^\star P(\rho), \quad e_0(\vec{r},t) = \rho - \sum_{q=1}^{Q-1} e_q^+, \quad P = c_s^2 \rho,$$
$$e_q^-(\vec{r},t) = J_q^\star - \frac{F_q^\star}{\lambda^-} = j_q^\star + \Lambda_o F_q^\star, \quad \vec{j} = \vec{J} + \frac{1}{2}\vec{F}, \quad \forall \quad \vec{F},$$
$$J_q^\star = t_q^\star(\vec{J}\cdot\vec{c}_q), \quad F_q^\star = t_q^\star(\vec{F}\cdot\vec{c}_q), \quad j_q^\star = t_q^\star(\vec{j}\cdot\vec{c}_q). \tag{8}$$

Here, c_s^2 is a free parameter and the isotropic weights $t_q^\star$ obey the constraints [8]

$$\sum_{q=1}^{Q-1} t_q^\star c_{q\alpha} c_{q\beta} = \delta_{\alpha\beta}, \quad \forall \quad \alpha,\beta, \quad 3\sum_{q=1}^{Q-1} t_q^\star c_{q\alpha}^2 c_{q\beta}^2 = 1, \quad \alpha \neq \beta. \tag{9}$$

The exact steady state conservation equations are

$$\sum_{q=1}^{Q-1} g_q^+(\vec{r}) = 0, \quad \sum_{q=1}^{Q-1} g_q^- \vec{c}_q(\vec{r}) = \vec{F}(\vec{r}). \tag{10}$$

With substitution of the recurrence relations (4) they yield

$$\sum_{q=1}^{Q-1} \bar{\Delta}_q e_q^- = \sum_{q=1}^{Q-1} \Lambda_o \Delta_q^2 e_q^+ - \left(\Lambda_{eo} - \frac{1}{4}\right)\sum_{q=1}^{Q-1} \Delta_q^2 g_q^+, \tag{11}$$

$$\sum_{q=1}^{Q-1} \bar{\Delta}_q e_q^+ \vec{c}_q = \vec{F} + \sum_{q=1}^{Q-1} \Lambda_e \Delta_q^2 e_q^- \vec{c}_q - \left(\Lambda_{eo} - \frac{1}{4}\right)\sum_{q=1}^{Q-1} \Delta_q^2 g_q^- \vec{c}_q. \tag{12}$$

Owing to the linearity of the equilibrium function (8) with respect to $P_q^\star(\vec{r})$, $j_q^\star(\vec{r})$, and $\Lambda_o F_q^\star(\vec{r})$, the postcollision solution can be written as a linear combination of one-argument functions, here $\gamma_q(\cdot)$ and $\Gamma_q(\cdot)$:

$$g_q^+(\vec{r}) = \gamma_q(j_q^\star) + \Lambda_o \gamma_q(F_q^\star) - 2\Lambda_o \Gamma_q(P_q^\star),$$
$$g_q^-(\vec{r}) = \gamma_q(P_q^\star) - 2\Lambda_e[\Gamma_q(j_q^\star) + \Lambda_o \Gamma_q(F_q^\star)]. \tag{13}$$

Multiplying Eq. (11) by Λ_e and substituting relations (8) for $e_q^\pm(\vec{r})$ and relations (13) for $g_q^\pm(\vec{r})$, the exact conservation equations become

$$\sum_{q=1}^{Q-1} \bar{\Delta}_q \Lambda_e j_q^\star = \Lambda_{eo} \sum_{q=1}^{Q-1} \Delta_q^2 P_q^\star - \left(\Lambda_{eo} - \frac{1}{4}\right)\Big(\sum_{q=1}^{Q-1} \Delta_q^2 \gamma_q(\Lambda_e j_q^\star) + \Lambda_{eo}\sum_{q=1}^{Q-1} \Delta_q^2 \gamma_q(F_q^\star) - 2\Lambda_{eo}\sum_{q=1}^{Q-1} \Delta_q^2 \Gamma_q(P_q^\star)\Big),$$

$$\sum_{q=1}^{Q-1} \bar{\Delta}_q P_q^\star \vec{c}_q = \vec{F} + \sum_{q=1}^{Q-1} \Delta_q^2 \Lambda_e j_q^\star \vec{c}_q + \Lambda_{eo}\sum_{q=1}^{Q-1} \Delta_q^2 F_q^\star \vec{c}_q - \left(\Lambda_{eo} - \frac{1}{4}\right) \times \Big(\sum_{q=1}^{Q-1} \Delta_q^2 \gamma_q(P_q^\star)\vec{c}_q - 2\sum_{q=1}^{Q-1} \Delta_q^2 \Gamma_q(\Lambda_e j_q^\star)\vec{c}_q - 2\Lambda_{eo}\sum_{q=1}^{Q-1} \Delta_q^2 \Gamma_q(F_q^\star)\vec{c}_q\Big). \tag{14}$$

When $\Lambda_{eo} \equiv \frac{1}{4}$, the last terms vanish in relations (4), (11), and (12), then the nonequilibrium components and the conservation relations can be expressed via the variations of the equilibrium components. In the general case, substituting relations (13) into the recurrence equations (4) and (5), it appears that the solutions for the functions $\gamma_q(\cdot)$ and $\Gamma_q(\cdot)$ depend on the eigenvalues only via Λ_{eo} (see [27]). It follows that the macroscopic solutions of Eqs. (14) for $\Lambda_e \vec{j}(\vec{r})$ and $P(\vec{r})$ are independent of Λ_e and Λ_o, provided that their combination Λ_{eo} takes a constant value and $\vec{F}$ is either independent of the eigenvalues (e.g., a constant forcing) or depends on them via Λ_{eo} [e.g., a resistance forcing below, proportional to $\Lambda_e \vec{j}(\vec{r})$]. This property appears to be sufficient to explain the parametrization role of the TRT eigenvalue combination Λ_{eo} for steady Stokes and dimensionless Navier-Stokes solutions in bulk. The TRT model is able then to yield viscosity-independent numerical errors for steady solutions. The necessary condition is to keep Λ_{eo} at a fixed value with the help of the free eigenvalue λ^- when the kinematic viscosity $\nu = \frac{1}{3}\Lambda_e$ varies.

C. Permeability measurements with the TRT and BGK models

It follows from the analysis of the Stokes equation that the components of the permeability tensor derived from the mean macroscopic velocity value via Darcy's law,

$$\nu\overline{\vec{j}} = \mathbf{K}\overline{(\vec{F} - \vec{\nabla} P)}, \quad \nu = \frac{1}{3}\Lambda_e, \tag{15}$$

may keep the same values for any ν provided that Λ_{eo} is set and the possible corrections due to boundary closure relations do not modify the parametrization properties of the bulk solutions. A boundary scheme is not guaranteed *a priori* to maintain the parametrization property of the evolution operator, e.g., the linear and quadratic interpolations [35] do not yield it (see the results in Table V in [28], Table 1 and Fig. 5, 6 in [29], and the analysis and Table VII in [26]). It is shown in [26] that the bounce-back, multireflection MR1 scheme [28] and several other classes of multireflection-type velocity and pressure schemes [26] keep the parametrization property. As a simplest example, locating the solid boundaries in the middle of the cut links ($\delta=\frac{1}{2}$), the relative error of the bounceback condition for the cubic law in the channel of width H is

$$\frac{k - k^{\text{th}}}{k^{\text{th}}} = \frac{\left(\frac{16}{3}\Lambda_{eo} - 4\delta^2\right)}{H^2}, \quad k^{\text{th}} = \frac{H^2}{12}. \tag{16}$$

Taking $\Lambda_{eo}=\frac{3}{16}$ one gets the exact value k^{th}. The solutions $\Lambda_{eo}=\frac{3\delta^2}{4}$ extend these results for any distance $\delta\vec{c}_q$ to the solid wall in a straight channel using the magic linear schemes from the (MGLI) family [26,36], which improve the linear interpolations for the parametrization property. This solution becomes $\Lambda_{eo}=\frac{3\delta^2}{2}$ for diagonal flow [26,37].

It follows that there is no one magic Λ_{eo} value for any flow. Moreover, if the numerical errors come only from the lack of accuracy of the boundary schemes for the parabolic velocity profiles, the macroscopic equations (14) contain the higher-order corrections for the general flows. The dependency of the permeability errors on the selected Λ_{eo} values is examined for a cubic arrays of spheres in [26] (Table VI). It is found that the MGLI schemes (which can be applied in a local form for stationary problems) are sufficiently accurate for porous flow when Λ_{eo} is smaller (roughly) than $\frac{1}{2}$.

The benefit of using a constant value of Λ_{eo} over the BGK model is illustrated for the permeability measurements of fibrous materials (Table III in [28]), a body-centered cubic array of spheres, and a random-sized sphere pack (Table III and Figs. 4 and 8 in [29]). The BGK model yields $\Lambda_{eo}=9\nu^2$, lacking any possibility of keeping Λ_{eo} constant when the viscosity varies. Using the bounceback condition for the cubic law, the error increases as $(48\nu^2-1)$. Small viscosity values are more accurate but quite inefficient for convergence to the steady state.

III. THE TRT MODEL FOR THE STOKES-BRINKMAN FLOW

Based on the equilibrium function (8), the Stokes-Brinkman LBE model incorporates the resistance force $\vec{F}=-\frac{\nu}{\nu_e}\phi\mathbf{K}^{-1}\nu_e\vec{j}$, where ϕ is the porosity, $\mathbf{K}$ is the permeability tensor of the modeled porous medium, and ν is the kinematic viscosity of the fluid. The parameter of the effective viscosity ν_e is $\frac{1}{3}\Lambda_e$ based on the second-order Chapman-Enskog analysis without accounting for variation of the forcing. Equation (14) present the exact TRT form of the modeled Brinkman equation (1) for the averaged velocity $\vec{u}=\frac{\vec{j}}{\rho_0}$ and the averaged pressure $\bar{P}=P(\rho)/\phi$. These equations indicate that $\nu_e\vec{j}$ is controlled by the nondimensional parameters, such as the porosity ϕ, the viscosity ratio $\frac{\nu}{\nu_e}$, and the Darcy number Da $=k/L^2$ (with L as a characteristic length for the isotropic mediim, $\mathbf{K}=k\mathbf{I}$), provided that Λ_{eo} is kept at a fixed value and $\vec{j}$ is set equal to $\vec{J}+\frac{1}{2}\vec{F}$. They suggest that the additional corrections may appear from the second- and higher-order variation of the forcing, described by $\Lambda_{eo}\Sigma_{q=1}^{Q-1}\Delta_q^2 F_q^{\star}\vec{c}_q$ and, except when $\Lambda_{eo}=\frac{1}{4}$, the last term in the momentum equation (14). Let us illustrate this on the solutions for the Stokes-Brinkman flow in a channel parallel to the arbitrarily inclined x' axis:

$$-F_{x'} = \rho_0\nu_e\partial_{z'}^2 u_{x'}, \quad \rho \equiv \rho_0, \quad \vec{F} = -F_c\vec{j}, \; F_c = \frac{\phi\nu}{k},$$

$$\vec{u} = \frac{\vec{j}}{\rho_0}, \quad \text{and} \quad \vec{j} = \frac{2\vec{J}}{2 + F_c}. \tag{17}$$

Hereafter, we work in the rotated coordinate system $\vec{\psi}_{\alpha'}=(\vec{\psi}\cdot\vec{1}_{\alpha'})$, $\forall\ \vec{\psi}$ and $\alpha'=\{x',z'\}$,

$$x' = x\cos\alpha + z\sin\alpha, \quad z' = -x\sin\alpha + z\cos\alpha, \tag{18}$$

and define the finite-difference type operators $\bar{\Delta}_{z',q}$ and $\Delta_{z',q}^2$ along the link $(\vec{c}_q, \vec{c}_{\bar{q}})$ as

$$\bar{\Delta}_{z',q}\psi(z') = \frac{\psi(z_q'^{+}) - \psi(z_q'^{+})}{2\Theta_q}, \tag{19}$$

$$\Delta_{z',q}^2\psi(z') = \frac{\psi(z_q'^{+}) - 2\psi(z') + \psi(z_q'^{+})}{\Theta_q^2},$$

$$\Theta_q = c_{qz'}, \quad c_{qz'} \neq 0, \tag{20}$$

where $|\Theta_q|$ is the distance along the z' axis between a grid node $\vec{r}$ and its grid neighbors $\vec{r}\pm\vec{c}_q$, their z' coordinates being denoted z' and $z_q'^{\pm}$, respectively.

A. Apparent viscosity in simply oriented channel flows

"Simple" flows are solutions in either a parallel (noninclined, $\alpha=0°$) or a diagonal ($\alpha=45°$) channel. Only the links with $c_{qx'}c_{qz'}\neq 0$ contribute then to the variation of $\vec{j}=j_{x'}(z')$. Combining the periodic condition $g_q^{\pm}(z')=0$ if $c_{qx'}c_{qz'}=0$ with the momentum constraint (10), here

$$\sum_{q=1}^{Q-1} g_q^-(z')c_{qx'} = F_{x'}, \quad \sum_{q=1}^{Q-1} g_q^-(z')c_{qz'} = 0, \tag{21}$$

the solution for $g_q^-(z')$ for the considered velocity sets is necessarily

$$g_q^-(z') = 3c_{qz'}^2 F_q^\star(z'), \quad \forall \quad q. \tag{22}$$

Substituting $F_q^\star = -t_q^\star F_c j_{x'}(z') c_{qx'}$, $\Delta_q^2 = \Theta_q^2 \Delta_{z',q}^2$, and relation (22) into Eq. (12), one gets

$$-F_{x'}(z') = (\Lambda_e - \Lambda_{eo} F_c) \sum_{q=1}^{Q-1} t_q^\star c_{qz'}^2 c_{qx'}^2 \Delta_{z',q}^2 j_{x'}(z') + 3 \times \left(\Lambda_{eo} - \frac{1}{4}\right) F_c \sum_{q=1}^{Q-1} t_q^\star \Theta_q^2 c_{qz'}^2 c_{qx'}^2 \Delta_{z',q}^2 j_{x'}(z'). \tag{23}$$

When $\alpha = 0°$ then $z^\pm = z \pm 1$ and $\Theta_q^2 = \Theta^2 = 1$ for all the nonhorizontal links. When $\alpha = 45°$ then $z'^\pm = z' \pm \frac{\sqrt{2}}{2}$ and $\Theta_q^2 = \Theta^2 = \frac{1}{2}$ for all the links with $c_{qx'} c_{qz'} \neq 0$. Replacing $\Delta_{z',q}^2$ with $\Delta_{z',\Theta}^2$ and Θ_q with Θ, and using the second property (9), the equivalent (exact) finite-difference form of the TRT momentum equation (23) is

$$\Delta_{z',\Theta}^2 j_{x'}(z') = 4b^2 j_{x'}(z') \quad \text{where} \quad b^2 = \frac{F_0}{4(1+\bar{\delta}\nu_e)},$$

$$F_0 = \frac{F_c}{\nu_e}, \quad \nu_e = \frac{1}{3}\Lambda_e, \tag{24}$$

$$\bar{\delta}\nu_e = \left[\Theta^2\left(\Lambda_{eo} - \frac{1}{4}\right) - \frac{\Lambda_{eo}}{3}\right] F_0. \tag{25}$$

Hereafter, $\nu_a = \nu_e + \delta\nu_e = \nu_e(1+\bar{\delta}\nu_e)$ and $\delta\nu_e = \nu_e \bar{\delta}\nu_e$ denote the apparent viscosity coefficient and its difference from the predicted value ν_e, respectively. The TRT numerical solution and its apparent viscosity are controlled then by two parameters F_0 and Λ_{eo}. The correction $\delta\nu_e$ vanishes if $\Lambda_{eo} = \frac{3}{8}$ and $\Lambda_{eo} = \frac{3}{4}$, for the parallel and the diagonal flow, respectively. The relation (A11) in Appendix A 2 extends the solution (25) for anisotropic force weights. Using them, one can also annhilate the numerical correction in the viscosity coefficients, at least for simply oriented channels as given by relation (A12).

The BGK scheme yields $\Lambda_{eo} = 9\nu^2$; then

$$\bar{\delta}\nu_e = \left(3\nu^2(3\Theta^2 - 1) - \frac{\Theta^2}{4}\right) F_0, \tag{26}$$

or, substituting $\nu_e = \frac{2\tau - 1}{6}$, equivalently,

$$\delta\nu_e = \begin{cases} F_c \dfrac{8\tau^2 - 8\tau - 1}{12}, & \alpha = 0°, \ \Theta^2 = 1, \\ F_c \dfrac{2\tau^2 - 2\tau - 1}{12}, & \alpha = 45°, \ \Theta^2 = \dfrac{1}{2}. \end{cases} \tag{27}$$

These solutions correspond to the formulas (11) and (12) of Nie and Martys [7], if we take there $c_s^2 = \frac{1}{3}$, $\delta t = 1$ and replace $\frac{\epsilon\nu}{k}$ with F_c. Their solutions are obtained by solving the BGK evolution equation with respect to velocity, with the help of the methodology [38]. The relation (26) shows that the numerical solution of the BGK scheme is not controlled by the governing parameter F_0 because of the dependency of the relative viscosity error $\bar{\delta}\nu_e$ on the viscosity. It is noted that the nonlinear function (26) increases rapidly with ν, with a prefactor equal to $(6\nu^2 - \frac{1}{4})$ in the straight channel.

B. Macroscopic solutions

The exact solution of the model Eq. (17) is

$$j_{x'}(z') = k_1 e^{2Bz'} + k_2 e^{-2Bz'}, \quad B = \frac{\sqrt{F_0}}{2}, \quad F_0 = \frac{F_c}{\nu_e}, \tag{28}$$

where the coefficients k_1 and k_2 are fixed by the boundary conditions. The solution of the numerical scheme (24) has the form

$$j_{x'}(z') = k_1 r^{z'} + k_2 r^{-z'}, \tag{29}$$

where

$$r^\Theta = s, \quad s = \frac{1+P}{1-P}, \quad P^2 = \frac{b^2\Theta^2}{1+b^2\Theta^2}. \tag{30}$$

Here, s is a root of the equation $s^2 - 2s(2b^2\Theta^2 + 1) + 1 = 0$. This numerical solution is equal to the exact solution (28) when $r = e^{2B}$, i.e.,

$$P = \tanh(B\Theta). \tag{31}$$

One satisfies this condition by either taking $F_0^{\text{num}} \neq F_0$,

$$F_0^{\text{num}}(B, \Lambda_{eo}, \Theta) = \frac{12}{4\Lambda_{eo}(1 - 3\Theta^2) + 3\Theta^2 \coth^2(\Theta B)}, \quad \forall \quad \Lambda_{eo}, \tag{32}$$

or, alternatively, prescribing

$$\Lambda_{eo}(B,\Theta) = \frac{3[1 - B^2\Theta^2 \coth^2(\Theta B)]}{4B^2(1 - 3\Theta^2)} \quad \text{when} \quad F_0^{\text{num}} = F_0 = 4B^2. \tag{33}$$

The TRT model can satisfy the condition (33) for any pair of prescribed values F_c^{num} and ν_e^{num} such that $\Lambda_{eo}(B,\Theta) > 0$, $B^2 = \frac{1}{4}(\frac{F_c}{\nu_e})^{\text{num}}$. For the BGK operator $\nu_e^{\text{num}} = \frac{1}{3}\sqrt{\Lambda_{eo}(B,\Theta)}$, then F_c^{num} is fixed by the selected B value.

Numerical validations of the exact solutions obtained above and in Appendixes A 2 and A 3 need to prescribe exactly the incoming populations $f_{\bar{q}}(\vec{r}_b) = (e_{\bar{q}} + \frac{g_q^+}{\lambda^+} - \frac{g_q^-}{\lambda^-})(\vec{r}_b)$ at the grid boundary nodes $\vec{r}_b$. When the assumed solution satisfies the evolution equation exactly, the system will converge to it when the exact boundary closure relations are used (or stay on it when the exact populations are intialized). This was validated for the parallel and diagonal flow using the $d3Q15$ velocity set and prescribing the solution (22) and (A2) for the nonequilibrium components. When the nonequilibrium solution is not known, one can take $\Lambda_{eo} = \frac{1}{4}$ and compute $g_q^\pm$ in the finite-difference form given by relations (4) on the assumed equilibrium solution. Moreover, given the macroscopic boundary values $\vec{u}^b(\vec{r}_b)$, the linear combination

$$f_{\bar{q}}(\vec{r}_b,t+1)=[\tilde{f}_q-\tilde{f}_{\bar{q}}+4\Lambda_o(g_q^- - F_q^\star)](\vec{r}_b,t)+\tilde{f}_q(\vec{r}_b-\vec{c}_q,t) - 4t_q^\star\rho_0[\vec{u}^b(\vec{r}_b)\cdot\vec{c}_q] \quad (34)$$

yields the exact closure relation $j_q(\vec{r}_b)=\rho_0[\vec{u}^b(\vec{r}_b)\cdot\vec{c}_q]$ for any steady solution. This scheme represents the MR1 scheme [28,36] for $\delta_q=0$, suitable only when the flat walls are located at the grid vertices. However, the scheme (34) conserves some specific nonequilibrium distributions (see [36]) and it should be restricted to testing of the derived solutions, e.g., based on the exact initialization of the populations.

C. Arbitrarily oriented flow

The solution (22), $g_q^-(z')=3c_{qz'}^2F_q^\star$, is valid for arbitrarily rotated force-driven Poiseuille flow. Then $g_q^-(z')$ is constant along each link, $\Delta_q^2 g_q^-=0$, and the directional Laplace operators $\Delta_{z',q}^2 j_{x'}(z')$ in Eq. (23) reduce to $\partial_{z'}^2 j_{x'}$, giving the exact Poiseuille profile. However, except for the linear and parabolic flows, $\Delta_{z',q}^2 j_{x'}(z')$ differ for all nonparallel links. We conjecture, analyzing the possible projections of $\{g_q^-\}$ on the relevant antisymmetric basis vectors [e.g., given by relations (A7)], that the solution in the form (29), $j_{x'}(z')=k_1r^{z'}+k_2r^{-z'}$, cannot satisfy the reference equations (4) in an arbitrarily rotated channel, at least using the isotropic force weights. However, we could show that this solution can be obtained for the (scalar) variable $\psi(z')$ of the diffusion equation $F_0\psi=\partial_{z'}^2\psi$, solving it with the L model and the anisotropic set of antisymmetric relaxation rates. The hydrodynamic modeling is restricted to the isotropic linkwise operators (TRT model) because of the additional momentum constraint. Let us then limit ourselves to the second-order (isotropic) approximation

$$\Delta_{z',q}^2 j_{x'}(z') \approx \partial_{z'}^2 j_{x'}(z') \quad \text{when } c_{qx'}c_{qz'}\neq 0. \quad (35)$$

Moreover, assuming that the solution (22) presents, owing to the momentum properties (21), the second-order approximation for $g_q^-(z')$ in arbitrarily rotated flow, the momentum equation keeps the form (23) and yields

$$-F_{x'}(z') \approx \nu_a^{(2)}\partial_{z'}^2 j_{x'}(z'), \quad \nu_a^{(2)}=\nu_e(1+\bar{\delta}^{(2)}\nu_e),$$

$$\bar{\delta}^{(2)}\nu_e=\left\{\bar{\nu}_\alpha\left(\Lambda_{eo}-\frac{1}{4}\right)-\Lambda_{eo}\left[\frac{1}{3}+\left(\frac{1}{3}-\bar{\nu}_\alpha\right)k_{x'}\right]\right\}F_0. \quad (36)$$

Here, $\bar{\nu}_\alpha$ is the model parameter given by relation (A8) and $k_{x'}\neq 0$ corresponds to the anisotropic force weights (A6). For simple orientations, $\bar{\nu}_\alpha=\Theta^2$ and $\bar{\delta}^{(2)}\nu_e$ reduces to $\bar{\delta}\nu_e$, given by relation (A11) and then to relation (25) for isotropic force weights ($k_{x'}=0$). One can again remove $\bar{\delta}^{(2)}\nu_e$ with the help of Λ_{eo} or $k_{x'}$, but only for each particular orientation. The solution (36) shows that the *second-order* estimate of the difference between the apparent and predicted values, $\nu_a^{(2)}-\nu_e$, is equal to $-\frac{\Lambda_{eo}F_c}{3}$ when $\Lambda_{eo}=1/4$ and $k_{x'}=0$. This value also follows from the second-order Chapman-Enskog analysis, when the second-order force gradients are taken into account, but (as usual) the second-order variation of g_q^- [the term $-\Sigma_{q=1}^{Q-1}(\Lambda_{eo}-\frac{1}{4})\Delta_q^2 g_q^-\vec{c}_q$ in Eq. (12)] is neglected. For the Brinkman flow $\Delta_q^2 g_q^-\propto\Delta_q^2 j_{x'}$ and the omission of this term becomes inconsistent, except for $\Lambda_{eo}=\frac{1}{4}$.

We suggest keeping Λ_{eo} close to $\frac{1}{4}$ when one can expect significant corrections from the variation of the nonequilibrium components and/or source terms. Using the standard equilibrium weights ($k_{x'}=0$), one can consider then $\nu_a^{(2)}=\nu_e(1-\frac{\Lambda_{eo}}{3}F_0)$ as the apparent (isotropic) viscosity coefficient. The possibility of reducing the next-order correction is considered in Appendix A 3. However, further numerical work is needed to establish optimal and robust strategies for realistic computations.

D. Navier-Stokes equilibrium

We recall that, for simply oriented channel flow, the macroscopic velocity solution of the TRT Brinkman model based on the linear equilibrium (8) satisfies the finite-difference type equations (24) and (25). These relations are also valid for the arbitrarily rotated Stokes-Poiseuille flow, which is the exact solution of the linear LBE schemes with $\bar{\delta}\nu_e\equiv 0$. We examine in Appendix B the possibilities of getting a one-dimensional solution $\vec{j}=j_{x'}(z')$ when the symmetric equilibrium component gets the nonlinear term $E_q^+(\vec{j},\hat{\rho},\alpha_\epsilon)$, given by relation (B1) as a function of the free equilibrium parameter α_ϵ. We keep in mind $\hat{\rho}=\rho_0$ for the incompressible Navier-Stokes equation and $\hat{\rho}=\phi\rho_0$ for the LBE modeling of the Forchheimer-Brinkman equation (e.g., [20]).

It is found that the velocity solutions for Poiseuille and Brinkman flows remain unchanged in the parallel channel for any α_ϵ, but the population solution gets corrections that conserve both mass and momentum and are given by relations (B6) and (B14). This solution is maintained for the Poiseuille flow by the principal linkwise boundary schemes, like bounceback, linear interpolations, and several others (see Sec. 3.2 in [36]).

However, a one-dimensional diagonal velocity is compatible with the mass and momentum constraints only for one particular value α_ϵ, given by the solution (B13), for both Poiseuille and Brinkman flow. This choice differentiates the nonlinear term $E_q^+(\vec{j},\hat{\rho},\alpha_\epsilon)$ from the standard one [8]. The diagonal flow has then equal solutions for Stokes and Navier-Stokes equilibrium using the BGK, TRT, or multiple-relaxation-time (MRT) schemes.

The arbitrarily rotated parabolic flow is the solution of the LBE schemes in bulk only when α_ϵ satisfies relation (B13) and $\Lambda_{eo}=\frac{1}{12}$. Exact numerical validation of this solution is possible, e.g., by using the exact population solution for all incoming populations, $f_{\bar{q}}(\vec{r}_b)=(e_{\bar{q}}+\frac{g_q^+}{\lambda^+}-\frac{g_q^-}{\lambda^-})(\vec{r}_b)$, where $\{g_q^\pm\}$ is given by relations (22) and (A2) plus the correction (B6). No exact solution for Stokes or Navier-Stokes Brinkman scheme was found in an arbitrarily rotated channel.

IV. COEFFICIENTS OF THE CHAPMAN-ENSKOG EXPANSION FOR STEADY SOLUTIONS

Nie and Martys [7] interpret the angular-dependent deviation $\delta\nu_e$ of the apparent viscosity coefficient from the predicted one as a "breakdown" of the Chapman-Enskog (*second-order*) expansion. Our goal is to show that if the solution exists as a *full* (infinite) Chapman-Enskog series, it then satisfies the reference equations and their macroscopic relations. The full series yields then the exact solution for the apparent viscosity for the Brinkman flow. A key point is to put again the force quantities $-F_q^{\star}/\lambda^-$ into e_q^-; one gets then with no extra effort the gradients of the forcing in the expansion, together with the ones for the momentum. Otherwise, it is risky to truncate a variation of the forcing, as for the second-order Chapman-Enskog analysis [7,19,20]. Following the principal idea of the Chapman-Enskog method, let us assume that the steady nonequilibrium solution of the L model can be expanded around the local equilibrium:

$$f_q^{\pm}(\vec{r}) = e_q^{\pm}(\vec{r}) + n_q^{\pm}(\vec{r}),\ n_q^{\pm}(\vec{r}) = \sum_{l\geq 1} n_q^{\pm}(\vec{r}),\ \text{then}$$

$$g_q^{\pm}(\vec{r}) = \sum_{l\geq 1} g_q^{\pm(l)}(\vec{r}),\ g_q^{\pm(l)} = \lambda_q^{\pm} n_q^{\pm(l)}. \quad (37)$$

With the help of the directional derivatives

$$\partial_q^k \psi = (\vec{c}_q \cdot \vec{\nabla})^k \psi,\ \forall \psi,\ k \geq 1, \quad (38)$$

we look for the infinite solution in the form

$$g_q^{\pm(2k-1)}(\vec{r}) = \frac{a_{2k-1}\partial_q^{2k-1} e_q^{\mp}(\vec{r})}{(2k-1)!}, \quad k \geq 1,$$

$$g_q^{\pm(2k)}(\vec{r}) = -2\Lambda_q^{\mp} \frac{a_{2k}\partial_q^{2k} e_q^{\pm}(\vec{r})}{(2k)!}, \quad k \geq 1. \quad (39)$$

The coefficients $\{a_{2k-1}, a_{2k}\}$, $k \geq 1$, are derived in Appendix C:

$$a_1 = 1,\ a_2 = 1,$$

$$a_{2k-1} = 1 + 2\left(\Lambda_q^{eo} - \frac{1}{4}\right) \sum_{1\leq n<k} a_{2n-1} \frac{(2k-1)!}{(2n-1)!\,[2(k-n)]!},$$

$$k \geq 2,$$

$$a_{2k} = 1 + 2\left(\Lambda_q^{eo} - \frac{1}{4}\right) \sum_{1\leq n<k} a_{2n} \frac{(2k)!}{(2n)!\,[2(k-n)]!}, \quad k \geq 2. \quad (40)$$

The solution (37) and (39) with the coefficients (40) coincides with the solution [27] of the recurrence equations for $g_q^{\pm}(e_q^{\pm})$ assuming it in the form of an expansion around the equilibrium (see also Appendix C). The apparent macroscopic equations given by the infinite Chapman-Enskog expansion are then the same as those given by the solution of the recurrence equations, at least when the solution has the form of a series, and the possible corrections owing to closure boundary relations are not taken into account (see [27]). The second-order expansion in terms of $\varepsilon = \frac{1}{L}$, L being the characteristic size, corresponds to $l=2$ $(k=1)$ and yields for the TRT operator

$$g_q^{\pm(1)}(\vec{r}) = \partial_q e_q^{\mp}(\vec{r}), \quad n_q^{\pm(1)}(\vec{r}) = \frac{\partial_q e_q^{\mp}(\vec{r})}{\lambda^{\pm}};$$

then

$$g_q^{\pm(2)}(\vec{r}) = -\Lambda^{\mp}\partial_q^2 e_q^{\pm}(\vec{r}). \quad (41)$$

Substituting $e_q^-(z') = (1 - F_c\Lambda_o) j_q^{\star}(z')$ and assuming uniform pressure, the truncated momentum equation $\sum_{q=1}^{Q-1}(g_q^{-(1)} + n_q^{-(2)})\vec{c}_q = \vec{F}$ reduces to Eq. (36) with $\nu_a^{(2)} = \nu_e(1 - \frac{\Lambda_{eo}}{3}F_0)$, which corresponds there to $\Lambda_{eo} = \frac{1}{4}$, $k_{x'} = 0$.

V. CONCLUDING REMARKS

This paper extends the analysis of the exact parametrization properties of the numerical solutions of the hydrodynamic equations to the Stokes-Brinkman case. As for the Stokes and Navier-Stokes equations, only when the "irrelevant" collision eigenvalues are selected with specific rules, is the physically sound parametrization of the solutions kept for all orders. The analytical solutions for channel flows confirm that, when the forcing factor F_c and the parameter of effective viscosity ν_e vary but their ratio is kept at a fixed value, the TRT Brinkman evolution operator obtains the same solutions for the velocity provided that the eigenvalue combination Λ_{eo} is fixed to some value. Taking Λ_{eo} inside the interval $]0, \frac{1}{2}[$ avoids large bulk and boundary discretization errors. At the same time, the relative apparent correction to the viscosity coefficient is related linearly to ν_e^2 for the BGK-based Brinkman schemes and the obtained velocity is not fixed by F_c/ν_e, as the BGK-Stokes velocity solution is not fixed by the applied ratio of the forcing to kinematic viscosity.

One should keep in mind that the derived solutions (25) and (A11) for the apparent viscosity coefficient and the solutions (31)–(33) or (A12), which equate the TRT solutions either to the exact or to the finite-difference ones, are valid only for simply oriented Brinkman channel flows. The approximate solution (36) for the apparent viscosity allows us to extend them for arbitrary flows. The particular choice $\Lambda_{eo} = \frac{1}{4}$ avoids impact of the second- and higher-order variations of the nonequilibrium components and sources on the apparent transport coefficients, for any steady problem. The second-order estimate of the apparent relative correction to the viscosity coefficient reduces then to $\bar{\delta}\nu_e = -\frac{\Lambda_{eo}}{3}F_0$. One can remove this correction by subtracting $\Lambda_{eo}\Delta_q^2 F_q^{\star}$ from each population solution, for any forcing (Appendix A 3). It is noted that $\Lambda_{eo} = \frac{1}{4}$ also has advanced stability properties (see [25]) but this value is not the most accurate one for the bounceback condition in simply oriented channels.

It was shown that the exact channel solutions are not compatible with the nonlinear (Navier-Stokes) equilibrium term even for the parabolic (Poiseuille) flow, with a few exceptions. The first one is the parallel (noninclined) flow. The second one is the diagonal flow with a very particular choice of the equilibrium weights. This is also valid for the Brinkman flow. The arbitrarily rotated Poiseuille flow can become the exact solution of the LBE with Navier-Stokes equilibrium only when $\Lambda_{eo} = \frac{1}{12}$, in addition to the restriction obtained for the diagonal flow.

It was also shown that the infinite Chapman-Enskog steady expansion for the nonequilibrium solution components satisfies steady recurrence equations. The presented explicit solutions for the coefficients of the Chapman-Enskog expansion given by relations (40) allow any-order approximations of the stationary conservation relations to be built. The analysis based on recurrence equations is suitable for any form of the equilibrium and source terms. They allow the exact dependency on the relaxation rates for macroscopic equations to be examined, without construction of exact microscopic or macroscopic solutions.

The results presented here were derived using the TRT model, but their extension to MRT models follows the same lines, using the recurrence equations derived in [27] for the most general MRT L model. Their sufficient parametrization condition is to maintain all nonzero magic combinations for the symmetric and antisymmetric modes at fixed values. We also expect that the analysis of the Navier-Brinkman MRT- or TRT-based models is straightforward, by adding the Forchheimer drag to the resistance force (see [20,40]) and combining the Navier-Stokes analysis [26,27] and the present approach. The recurrence equations for the transient solutions can be found in [27] but they have not yet been explored with respect to high-order Chapman-Enskog analysis.

The problem of the boundary conditions for the Brinkman schemes was not addressed in this paper, the numerical validation being restricted to the exact boundary closure relations. However, the form of the nonequilibrium solutions given for the simple flow by relations (22) and (A3) tells that their discrepancy from the "standard" solution (corresponding to a constant forcing) is proportional to the apparent viscosity correction $\bar{\delta}\nu_e$. This suggests a revision of the closure relations set by the bounceback, linear interpolation, and other schemes, with respect to their effective accuracy already at the first order. It would also be interesting to develop the interface analysis of the lattice Boltzmann Brinkman scheme for materials with a large contrast of permeability values.

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APPENDIX A: BRINKMAN SIMPLY ORIENTED CHANNEL FLOW: DETAILS

1. Continuity equation

The recurrence equations (5) yield

$$\Lambda_e \Delta_q^2 g_q^+(z') = (\Delta_q^2 e_q^+ - \bar{\Delta}_q g_q^-)(z'). \qquad \text{(A1)}$$

Let us prescribe the uniform solution for $e_q^+(z')$ and the solution (22) for $g_q^-(z')$. Substituting then $\Lambda_e \Delta_q^2 g_q^+(z') = -\bar{\Delta}_q g_q^-(z')$ into Eq. (4), the solution for $g_q^+(z')$ is given by

$$\Lambda_e g_q^+(z') = \Lambda_e \bar{\Delta}_q j_q^\star(z') + \left[\Lambda_{eo} - 3\Theta^2\left(\Lambda_{eo} - \frac{1}{4}\right)\right]\bar{\Delta}_q F_q^\star(z'), \qquad \text{(A2)}$$

or, equivalently,

$$g_q^+(z') = (1 + \bar{\delta}\nu_e)\bar{\Delta}_q j_q^\star(z'). \qquad \text{(A3)}$$

Taking the sum of relation (A3), the mass conservation equation (11) yields

$$\sum_{q=1}^{Q-1} g_q^+(z') = (1 + \bar{\delta}\nu_e)\bar{\Delta}_{z',\Theta} j_{x'}\Big(\sum_{q=1}^{Q-1} t_q^\star c_{qx'} c_{qz'}\Big) = 0. \qquad \text{(A4)}$$

In contrast to the momentum equation, the continuity equation does not get any corrections from variation of the forcing in simple channel flows. When $\bar{\delta}\nu_e$ vanishes (e.g., for a special choice of free parameters), g_q^+ takes the form that one would expect for the channel flow:

$$g_q^+(z') = \bar{\Delta}_q j_q^\star(z'), \qquad \text{(A5)}$$

which reduces to $g_q^+(z') = t_q^\star \partial_{z'} j_{x'}(z') c_{qx'} c_{qz'}$ for the parabolic flow.

2. Anisotropic force weights

Let us now examine the modification of the equilibrium force quantities in the form

$$F_q^\star \to F_q^\star + F_q^{\star a}, \quad F_q^{\star a} = k_{x'} F_{x'} h_{qx'} + k_{z'} F_{z'} h_{qz'},$$

$$h_{qx'} = t_q^\star c_{qx'}(1 - 3c_{qz'}^2), \quad h_{qz'} = t_q^\star c_{qz'}(1 - 3c_{qx'}^2). \qquad \text{(A6)}$$

Here $k_{x'}$ and $k_{z'}$ are some constants and $\{h_{qx'}\}$ and $\{h_{qz'}\}$ are orthogonal, mass- and momentum-conserving basis vectors of the MRT model:

$$\sum_{q=1}^{Q-1} h_{q\alpha'} = 0, \quad \sum_{q=1}^{Q-1} h_{q\alpha'} c_{q\beta'} = 0, \quad \forall \ \alpha', \beta'. \qquad \text{(A7)}$$

They obey the following properties, derived with the help of relations (9):

$$\sum_{q=1}^{Q-1} h_{q\alpha'} c_{q\alpha'} c_{q\beta'}^2 = \frac{1}{3} - \bar{\nu}_\alpha,$$

$$\bar{\nu}_\alpha = 3\sum_{q=1}^{Q-1} t_q^\star c_{q\alpha'}^2 c_{q\beta'}^4 = \frac{1}{4}(3 + \cos 4\alpha), \quad \forall \ \alpha' \neq \beta', \qquad \text{(A8)}$$

$$\sum_{q=1}^{Q-1} h_{q\beta'} c_{q\alpha'} c_{q\beta'}^2 = 0, \quad \forall \ \alpha' \neq \beta', \quad \sum_{q=1}^{Q-1} h_{q\alpha'} c_{q\alpha'}^3 = \frac{1}{2}\sin^2 2\alpha, \quad \forall \ \alpha', \qquad \text{(A9)}$$

$$\sum_{q=1}^{Q-1} h_{qz'}c_{qx'}^3 = -\frac{1}{2}\sin 4\alpha, \quad \sum_{q=1}^{Q-1} h_{qx'}c_{qz'}^3 = \frac{1}{2}\sin 4\alpha. \tag{A10}$$

These relations are valid for the $d2Q9$, $d3Q15$, and $d3Q19$ velocity sets. The solution (22) keeps its form, and the apparent correction $\bar{\delta}\nu_e$ given by relation (24) then gets the term $-\Lambda_{eo}(\frac{1}{3}-\bar{\nu}_\alpha)k_{x'}F_0$ [from the first term in the right-hand side (RHS) of Eq. (23), $\Sigma_{q=1}^{Q-1}\Delta_q^2\Lambda_e e_q^- \vec{c}_q$, using relations (A8)]. It becomes for simple orientations (where $\bar{\nu}_\alpha=\Theta^2$)

$$\bar{\delta}\nu_e = \left\{\Theta^2\left(\Lambda_{eo}-\frac{1}{4}\right) - \Lambda_{eo}\left[\frac{1}{3}+\left(\frac{1}{3}-\Theta^2\right)k_{x'}\right]\right\}F_0. \tag{A11}$$

The following solution annihilates it:

$$k_{x'} = k_0 \quad \text{where } k_0 = \frac{\frac{3}{4}\Theta^2-(3\Theta^2-1)\Lambda_{eo}}{\Lambda_{eo}(3\Theta^2-1)}, \quad \alpha=\{0°,45°\}. \tag{A12}$$

3. Further reduction of the apparent viscosity corrections

Let us examine what happens if one tries to remove the terms due to force variation from the macroscopic equation taking (for any forcing, in principle)

$$g_q^- \to g_q^- - \Lambda_{eo}\Delta_q^2 F_q^\star, \quad \Delta_q^2 F_q^\star = F_q^\star(\vec{r}+\vec{c}_q) - 2F_q^\star(\vec{r}) + F_q^\star(\vec{r}-\vec{c}_q), \quad q=1,\ldots,Q-1. \tag{A13}$$

One can include again the total momentum correction in e_q^- [cf. relation (8)]:

$$e_q^- = j_q^\star + \Lambda_o F_q^\star \to j_q^\star + \Lambda_o F_q^\star - I_l\Lambda_{eo}\Delta_q^2 F_q^\star. \tag{A14}$$

Here $I_l=-\frac{1}{\lambda^-}>0$ if $\vec{j}$ is defined with relation (17) or $I_l=\Lambda_o$ if $j_q^\star=J_q^\star+\frac{1}{2}F_q^\star$ includes $-\frac{1}{2}\Lambda_{eo}\Delta_q^2F_q^\star$ (see below). Substituting $g_q^-=-\Lambda_e\bar{\Delta}_q e_q^-$ [see Eqs. (4) when $\Lambda_{eo}=\frac{1}{4}$] with relation (A14) into the momentum equation (10), it takes the following (exact) form for simple flow:

$$\frac{A^2}{4}\Delta_{z',\Theta}^2(\Delta_{z',\Theta}^2 j_{x'}) + \Delta_{z',\Theta}^2 j_{x'} = F_0 j_{x'}, \quad A^2 = 4F_c I_l\Lambda_{eo}\Theta^2, \tag{A15}$$

when

$$\vec{F} = -F_c(\vec{j} - \Lambda_{eo}\Theta^2\Delta_{z',\Theta}^2\vec{j}). \tag{A16}$$

Looking again for $j_{x'}(z')$ in the form (29), and owing to the relation

$$\Delta_{z',\Theta}^2(\Delta_{z',\Theta}^2 r^{z'}) = 4a^2\Delta_{z',\Theta}^2 r^{z'} \quad \text{where } 4a^2 = \frac{r^\Theta - 2 + r^{-\Theta}}{\Theta^2}, \tag{A17}$$

Eq. (A15) is satisfied if

$$a^2 = \frac{-1+\sqrt{1+A^2F_0}}{2A^2}. \tag{A18}$$

The velocity solution satisfies again the finite-difference scheme (24) but with b^2 replaced there by a^2. With the relation (A18) substituted into Eq. (A15), an equivalent finite-difference equation takes the form

$$\Delta_{z',\Theta}^2 j_{x'} = 4a^2 j_{x'}, \quad a^2 = \frac{F_0}{4(1+\bar{\delta}^{(4)}\nu_e)} \quad \text{where } \bar{\delta}^{(4)}\nu_e = A^2a^2 = \frac{1}{2}(-1+\sqrt{1+A^2F_0}). \tag{A19}$$

This shows that removal of the principal force correction (A13) when $\Lambda_{eo}=\frac{1}{4}$ does not annihilate the apparent viscosity correction, because of the proportionality of $\Delta_{z',\Theta}^2(\Delta_{z',\Theta}^2 j_{x'})$ and $j_{x'}$. Indeed, the negative correction $\bar{\delta}^{(2)}\nu_e=-\frac{\Lambda_{eo}}{3}F_0$ is replaced here with the positive correction $\bar{\delta}^{(4)}\nu_e$, $\bar{\delta}^{(4)}\nu_e<|\bar{\delta}\nu_e|$, only when $\Lambda_{eo}F_0<\frac{3}{2}$ in parallel flow. When $\Lambda_{eo}F_0$ is small enough and $I_l=\Lambda_o$ then $\bar{\delta}^{(4)}\nu_e$ behaves as $\frac{A^2F_0}{4}=\frac{1}{3}(\Lambda_{eo}F_0)^2\Theta^2$, confirming the expected (anisotropic) behavior $(\Lambda_{eo}F_0)^n$ for higher-order corrections $\bar{\delta}^{(2n)}\nu_e$.

In agreement with the dimensional analysis above, the exact solution is controlled by F_0 and Λ_{eo} only when $I_l=\Lambda_o$, i.e., when $j_q^\star$ is related to $J_q^\star$ via one-half of the whole force quantity (A16). Using the relation (A19), one gets $\vec{j}=\frac{2\vec{j}}{2+F_c(1-4a^2\Lambda_{eo}\Theta^2)}$. When $\Lambda_{eo}=\frac{1}{4}$, $k_x=0$, and the correction (A13) is included, the truncated equation for arbitrary flow is given by relation (36) with $\bar{\delta}^{(2)}\nu_e\approx 0$, provided that F_0 is sufficiently small and the fourth- and higher-order corrections can be neglected.

APPENDIX B: POISEUILLE AND BRINKMAN FLOW BASED ON THE NONLINEAR EQUILIBRIUM FUNCTIONS

Let us add the nonlinear term $E_q^+(\vec{j},\hat{\rho},\alpha_\epsilon)$ to e_q^+ in relation (8),

$$E_q^+(\vec{j},\hat{\rho},\alpha_\epsilon) = t_q^\star\frac{3j_q^2-\|\vec{j}\|^2}{2\hat{\rho}} - \frac{\|\vec{j}\|^2}{\hat{\rho}}\alpha_\epsilon\epsilon_q. \tag{B1}$$

When $\alpha_\epsilon=0$, the distribution function (B1) takes its original form [8,39] (restricted there to $c_s^2=\frac{1}{3}$). The isotropic distribution $\{\epsilon_q=\epsilon_p^\star\}$ has one value per velocity class $p=\|\vec{c}_q\|^2$. This is given by the components of the fourth-order polynomial basis vectors of the MRT model, e.g., $\epsilon_0^\star=-4$, $\epsilon_1^\star=2$, $\epsilon_2^\star=-1$ for $d2Q9$, $\epsilon_0^\star=8$, $\epsilon_1^\star=-2$, $\epsilon_3^\star=1/2$ for $d3Q15$, and $\epsilon_0^\star=12$, $\epsilon_1^\star=-4$, $\epsilon_2^\star=1$ for $d3Q19$ (see [18,23,32,33]). The vector $\{\epsilon_q\}$ conserves the mass, $\Sigma_{q=0}^{Q-1}\epsilon_q=0$, and its second moments vanish, $\Sigma_{q=1}^{Q-1}\epsilon_q c_{q\alpha}c_{q\beta}=0$, $\forall\ \alpha,\beta$. The equilibrium correction $\{\alpha_\epsilon\epsilon_q\}$ does not influence then the second-order conservation equations; its possible impact on the stability was numerically investigated by Lallemand and Luo [33].

In the presence of the nonlinear term, the postcollision solution (13) gets the corrections $g_q^+(\vec{r})\to g_q^+(\vec{r})-2\Lambda_o\Gamma_q(E_q^+)$ and $g_q^-(\vec{r})\to g_q^-(\vec{r})+\gamma_q(E_q^+)$. Owing to the linearity of the recurrence equations with respect to the equilibrium components, the recurrence equations (4) yield

$$2\Gamma_q(E_q^+) = \Delta_q^2 E_q^+ + 2\left(\Lambda_{eo} - \frac{1}{4}\right)\Delta_q^2 \Gamma_q(E_q^+),$$

$$\gamma_q(E_q^+) = \Delta_q E_q^+ + \left(\Lambda_{eo} - \frac{1}{4}\right)\Delta_q^2 \gamma_q(E_q^+). \quad \text{(B2)}$$

The fourth-order accurate approximation of these equations takes the form

$$2\Gamma_q(E_q^+) = \Delta_q^2 E_q^+ + \left(\Lambda_{eo} - \frac{1}{4}\right)\Delta_q^2 \Delta_q^2 E_q^+, \quad \text{(B3)}$$

$$\gamma_q(E_q^+) = \Delta_q E_q^+ + \left(\Lambda_{eo} - \frac{1}{4}\right)\Delta_q^2 \Delta_q E_q^+. \quad \text{(B4)}$$

Let us assume now that the LBE system gets a one-dimensional invariant solution $\vec{j}=j_{x'}(z')$; then

$$E_q^+(\vec{j},\hat{\rho},\alpha_\epsilon) = -\frac{j_{x'}^2}{2\hat{\rho}} w_q, \quad w_q = 2\alpha_\epsilon \epsilon_q + t_q^\star(1 - 3c_{qx'}^2). \quad \text{(B5)}$$

The last terms in Eqs. (B3) and (B4) vanish for the linear velocity distribution, e.g., arbitrarily rotated Couette flow, where $\partial_{z'}^2 j_{x'}^2=\text{const}$. The solution given by relations (B3) and (B4) is exact for the parabolic flow where $\partial_{z'}^4 j_{x'}^2=\text{const}$, and they yield

$$\gamma_q(E_q^+) = \frac{1}{\hat{\rho}}[H^{-(1)}(z') + H^{-(3)}(z')] w_q c_{qz'} \quad \text{where } H^{-(1)} = j_{x'}\partial_{z'} j_{x'}, \;\; H^{-(3)} = 3\left(\Lambda_{eo} - \frac{1}{12}\right)(\partial_{z'} j_{x'} \partial_{z'}^2 j_{x'}) c_{qz'}^2,$$

$$-2\Lambda_o \Gamma_q(E_q^+) = -\frac{\Lambda_o}{\hat{\rho}}[H^{+(2)}(z') + H^{+(4)} \times (z')] w_q c_{qz'}^2 \quad \text{where } H^{+(2)} = j_{x'}\partial_{z'}^2 j_{x'} + (\partial_{z'} j_{x'})^2, \;\; H^{+(4)} = 3\left(\Lambda_{eo} - \frac{1}{6}\right)(\partial_{z'}^2 j_{x'})^2 c_{qz'}^2. \quad \text{(B6)}$$

This solution coincides with the exact Chapman-Enskog expansion for the parabolic flow [cf. relations (39) with (40) for $k=2$ or relations (3.6) in [36]]. The assumed one-dimensional parabolic profile may exist, but only provided that the expansion (B6) satisfies the conservation relations (10). The relevant moments are determined by the following relations, valid for the $d2Q9$, $d3Q15$, and $d3Q19$ models and any inclination angle α:

$$\sum_{q=1}^{Q-1} w_q = 0, \quad \sum_{q=1}^{Q-1} w_q c_{qz'} c_{qx'} = 0, \quad \sum_{q=1}^{Q-1} w_q c_{qz'}^2 = 0, \quad \text{(B7)}$$

$$\sum_{q=1}^{Q-1} w_q c_{qz'}^4 = -\frac{1}{2}\sin^2(2\alpha)(1 + k_\epsilon \alpha_\epsilon), \quad k_\epsilon = 24, \quad \text{(B8)}$$

$$\sum_{q=1}^{Q-1} w_q c_{qx'} c_{qz'}^3 = -3\sin(4\alpha)\alpha_\epsilon. \quad \text{(B9)}$$

It follows that the fourth-order corrections to the continuity equation and the momentum equations, along the z' and x' axes, respectively, become

$$-2\Lambda_o \sum_{q=1}^{Q-1} g_q^+(E_q^+) = \frac{-\Lambda_o H^{+(4)}}{\hat{\rho}} \sum_{q=1}^{Q-1} w_q c_{qz'}^4 = \frac{3\Lambda_o}{2\hat{\rho}}(\partial_{z'}^2 j_{x'})^2 \sin^2(2\alpha)(1 + k_\epsilon \alpha_\epsilon)\left(\Lambda_{eo} - \frac{1}{6}\right), \quad \text{(B10)}$$

$$\sum_{q=1}^{Q-1} g_q^-(E_q^+) c_{qz'} = \frac{H^{-(3)}}{\hat{\rho}} \sum_{q=1}^{Q-1} w_q c_{qz'}^4 = -\frac{3(\partial_{z'} j_{x'} \partial_{z'}^2 j_{x'})}{2\hat{\rho}} \sin^2(2\alpha) \times (1 + k_\epsilon \alpha_\epsilon)\left(\Lambda_{eo} - \frac{1}{12}\right), \quad \text{(B11)}$$

$$\sum_{q=1}^{Q-1} g_q^-(E_q^+) c_{qx'} = \frac{H^{-(3)}}{\hat{\rho}} \sum_{q=1}^{Q-1} w_q c_{qx'} c_{qz'}^3 = -\frac{9(\partial_{z'} j_{x'} \partial_{z'}^2 j_{x'})}{\hat{\rho}} \sin(4\alpha)\alpha_\epsilon \left(\Lambda_{eo} - \frac{1}{12}\right). \quad \text{(B12)}$$

These relations tell us that the corrections vanish for parallel parabolic one-dimensional flow ($\alpha=0°$). In the rotated flow, the corrections to the conservation equations vanish when the velocity field is linear, $\partial_{z'}^2 j_{x'}=0$. When the flow is diagonal ($\alpha=45°$), Eqs. (B12) have no corrections to the momentum equation for $j_{x'}(z')$ but the corrections (B10) to the continuity equation and the ones given by relations (B11) for the vertical momentum Eq. (B11) vanish only if

$$\alpha_\epsilon = -\frac{1}{Ck_\epsilon}, \quad k_\epsilon = 24 \quad \text{when } \epsilon_q = C\epsilon_p^\star, \quad \forall \; C. \quad \text{(B13)}$$

When α_ϵ satisfies relations (B13), then $E_q^+=0$ for all links with $c_{qz'}\neq 0$ in the diagonal flow. It follows that the postcollision correction (B6), describing its evolution, vanishes, and the nonequilibrium solution for both Poiseuille and Brinkman diagonal flows is the same as for the Stokes equilibrium and given by relations (22) and (A2). Finally, only when α_ϵ satisfies relation (B13) and $\Lambda_{eo}=\frac{1}{12}$ do all macroscopic corrections vanish for the arbitrarily rotated parabolic flow.

For Brinkman flows, the nonequilibrium correction satisfies the system (B2) and Eqs. (5). Computing E_q^+ with the velocity solution (29) and (30), the correction is simply $-2\Lambda_o\Gamma_q(E_q^+)=-\Lambda_o\Delta_q^2E_q^+$ and $\gamma_q(E_q^+)=\Delta_qE_q^+$ when $\Lambda_{eo}=\frac{1}{4}$. Equations (5) yield for arbitrary Λ_{eo}

$$-2\Lambda_o\Gamma_q(E_q^+)=-2\Lambda_o\frac{w_q}{\hat{\rho}}K(z)Rc_{qz}^2,\quad \gamma_q(E_q^+)=\frac{w_q}{\hat{\rho}}K(z)c_{qz},$$

$$K(z)=\frac{R}{(4\Lambda_{eo}R^2-1)}(k_1^2r^{2z}-k_2^2r^{-2z}),\quad R=\frac{(r^{2\Theta}-1)}{(r^{2\Theta}+1)}. \tag{B14}$$

The population solution in parallel Brinkman flow is given by relations (22) and (A2) plus the correction $\{-2\Lambda_o\Gamma_q(E_q^+),g_q^-(E_q^+)\}$. This correction has no impact on the macroscopic velocity. The diagonal Brinkman flow can take place only when $\alpha_\epsilon=-\frac{1}{Ck_\epsilon}$, $\epsilon_q=C\epsilon_p^\star$; the solution (22) and (A2) is then valid and the macroscopic solution is the same as for the Stokes-Brinkman modeling.

APPENDIX C: CHAPMAN-ENSKOG EXPANSION AND RECURRENCE EQUATIONS: DETAILS

Let us consider relations (37)–(39). We utilize the Chapman-Enskog methodology by applying the Taylor expansion to the spatial variation of the truncated solution $f_{(l),q}=e_q+\Sigma_{1\le i<l}n_q^{(i)}$, and a parity argument:

$$g_q^{\pm(2k-1)}(\vec{r})=\sum_{r=2s-1,1\le s\le k}\frac{\partial_q^r n_q^{\mp(2k-1-r)}}{r!}+\sum_{r=2s,1\le s<k}\frac{\partial_q^r n_q^{\pm(2k-1-r)}}{r!},$$

$$g_q^{\pm(2k)}=\sum_{r=2s,1\le s\le k}\frac{\partial_q^r n_q^{\pm(2k-r)}}{r!}+\sum_{r=2s-1,1\le s\le k}\frac{\partial_q^r n_q^{\mp(2k-r)}}{r!}\quad\text{with}\quad n_q^{\pm(0)}=e_q^{\pm}. \tag{C1}$$

Prescribing the form (39) for all the components $n_q^{\pm(i)}=\frac{1}{\lambda_q^\pm}g_q^{\pm(i)}$, the relations (C1) result in the four following relations:

$$\frac{a_{2k-1}}{(2k-1)!}=\frac{1}{(2k-1)!}+\frac{1}{\lambda_q^\pm}\sum_{1\le n<k}\frac{a_{2n-1}}{[2(k-n)]!(2n-1)!}-\frac{2\Lambda_q^\pm}{\lambda_q^\mp}\sum_{1\le n<k}\frac{a_{2n}}{[2(k-n)-1]!(2n)!}, \tag{C2}$$

$$-2\Lambda_q^\mp\frac{a_{2k}}{(2k)!}=\frac{1}{(2k)!}+\frac{1}{\lambda_q^\mp}\sum_{1\le n\le k}\frac{a_{2n-1}}{[2(k-n)+1]!(2n-1)!}-\frac{2\Lambda_q^\mp}{\lambda_q^\pm}\sum_{1\le n<k}\frac{a_{2n}}{[2(k-n)]!(2n)!}. \tag{C3}$$

We use the following equalities to derive from them relations (C5) and (C6):

$$\frac{2\Lambda_q^+}{\lambda_q^-}-\frac{2\Lambda_q^-}{\lambda_q^+}=\frac{1}{\lambda_q^+}-\frac{1}{\lambda_q^-}=\Lambda_q^--\Lambda_q^+,$$

$$\frac{1}{\lambda_q^\pm}-\frac{2\Lambda_q^\pm}{\lambda_q^\mp}=2\left(\Lambda_q^{eo}-\frac{1}{4}\right),\quad\text{and}\quad -\frac{2\Lambda_q^\mp}{\lambda_q^\pm}+\frac{4\Lambda_q^{eo}}{\lambda_q^\mp}=(-2\Lambda_q^\mp)\times\left[2\left(\Lambda_q^{eo}-\frac{1}{4}\right)\right]. \tag{C4}$$

With their help, equating the RHS of the two relations (C2), we get

$$\sum_{1\le n<k}\frac{a_{2n}}{(2n)![2(k-n)-1]!}=\sum_{1\le n<k}\frac{a_{2n-1}}{(2n-1)![2(k-n)]!}. \tag{C5}$$

Multiplying the upper and lower relations (C3) by $2\Lambda_q^+$ and $2\Lambda_q^-$, respectively, and equating then their RHSs, we obtain the second condition on the coefficients:

$$\sum_{1\le n\le k}\frac{a_{2n-1}}{(2n-1)![2(k-n)+1]!}=\frac{2}{(2k)!}+4\Lambda_q^{eo}\sum_{1\le n<k}\frac{a_{2n}}{(2n)![2(k-n)]!}. \tag{C6}$$

Replacing the last terms in all relations (C2) and (C3) with their equivalents given by relation (C5) and (C6), the coefficients $\{a_{2k-1},a_{2k}\}$ in relations (C2) become expressed via the previous-order coefficients *of the same parity*. This gives the solution (40).

It is straightforward to show that the obtained solution satisfies the recurrence equations (4), by substituting it into them and replacing there the lattice discrete operators by their Taylor approximation:

$$\bar{\Delta}_q\psi=\sum_{k\ge1}\frac{\partial_q^{2k-1}\psi}{(2k-1)!},\quad \Delta_q^2\psi=2\sum_{k\ge1}\frac{\partial_q^{2k}\psi}{(2k)!},\quad\forall\quad\psi. \tag{C7}$$

One gets with a parity argument from the recurrence equations (5)

$$2\bar{\Delta}_q\sum_{k\ge1}\frac{a_{2k}\partial_q^{2k}e_q^\mp}{(2k)!}=\Delta_q^2\sum_{k\ge1}\frac{a_{2k-1}\partial_q^{2k-1}e_q^\mp}{(2k-1)!},$$

$$\bar{\Delta}_q \sum_{k\geq 1} \frac{a_{2k-1}\partial_q^{2k-1} e_q^{\pm}}{(2k-1)!} = \Delta_q^2 e_q^{\pm} + 2\Lambda_q^{eo}\Delta_q^2 \sum_{k\geq 1} \frac{a_{2k}\partial_q^{2k} e_q^{\pm}}{(2k)!}. \tag{C8}$$

Expanding again the finite-difference linkwise operators into the series (C7), the relations (C8) become equivalent to relations (C5) and (C6). It follows that the obtained series satisfies the recurrence equations (4) and (5).

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